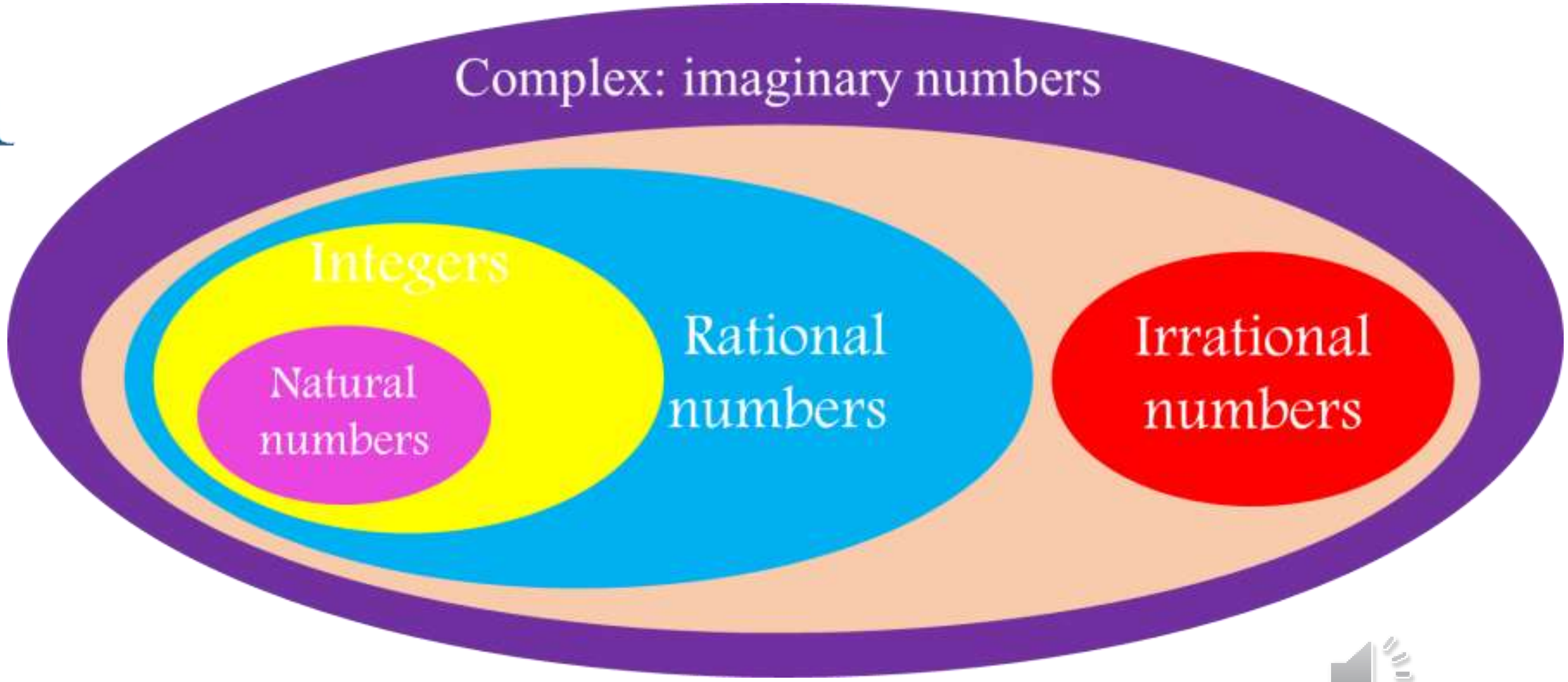
# *Complex Numbers*

## Part 1



# Introduction

$x^2 = -1$ No solution in IR

Suppose that there is a number $i$ such that:

$$i^2 = -1$$

$i$ is called imaginary number

# Definition

- ✓ A complex number z is in the form of $z = a + bi \;\; a, b \in IR$
- ✓ $\boldsymbol{a}$ is called the real part: $Re(z) = a$
  and $\boldsymbol{b}$ is called the imaginary part: $Im(z) = b$
- ✓ The set of all the complex numbers is denoted $\mathbb{C}$.
- ✓ Any real number is a complex number where 0 is its imaginary part.

Example:

$z = -1 - 3i$ where $Re(z) = -1$ and $Im(z) = -3$

$z = 2 + \frac{1}{2}i$ where $Re(z) = 2$ and $Im(z) = \frac{1}{2}$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❶ $z + z' = a + bi + a' + b'i = (a + a') + (b + b')i$

Example:

$z = 1 + 2i \quad ; \quad z' = 3 + 3i$

$z + z' = (1 + 4) + (3 + 3)i = 4 + 5i$

❷ $z - z' = a + bi - (a' + b'i) = (a - a') + (b - b')i$

Example:

$z = 1 + 2i \quad ; \quad z' = 3 + 3i$

$z - z' = (1 - 3) + (2 - 3)i = -2 + (-1)i = -2 - i$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❸ $z \times z' = (a + bi) \times (a' + b'i) = aa' - bb' + i(ab' + ba')$

Proof:

$z \times z' = (a + bi) \times (a' + b'i)$

$= aa' + ab'i + bia' + bib'i$

$= aa' + ab'i + ba'i + bb'i^2$

$= aa' + ab'i + ba'i - bb'$ since $i^2 = -1$

$= aa' - bb' + i(ab' + ba')$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❸ $z \times z' = (a + bi) \times (a' + b'i) = aa' - bb' + i(ab' + ba')$

Example:

$z = 1 + 2i \quad ; \quad z' = 3 + 3i$

First method: apply the above rule

$z \times z' = (1 + 2i)(3 + 3i) = 1 \times 3 - 2 \times 3 + i(1 \times 3 + 2 \times 3) = -3 + 9i$

Second method: multiply step by step

$z \times z' = (1 + 2i)(3 + 3i) = 3 + 3i + 6i + 6i^2 = 3 + 9i - 6 = -3 + 9i$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❹ If $z = 0$, then $a = b = 0$

Given $z = m - n + (n + 2)i$

Find the values of m and n so that z is null.

$m - n = 0$ and $n + 2 = 0$

$m = n$ $\qquad$ $n = -2$

$m = -2$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❺ If $z = z'$, then $a = a'$ and $b = b'$

Given $z = m - n + (n + 2)i$ and $z' = 3 + 3i$

Find the values of m and n so that z= $z$'.

$m - n = 3$ and $\quad n + 2 = 3$

$m = n + 3 \quad\quad n = 3 - 2 = 1$

$m = 1 + 3 = 4$

# Properties

Let $z = a + bi$ and $z' = a' + b'i$ be two complex numbers

❻ A complex number z is said to be pure imaginary if and only if $Re(z) = 0$ and $Im(z) \neq 0$.

Example:

$-\frac{1}{2}i$ ; $12i$ ; $i$ ; $-i$ ; ... are pure imaginary

Remark:

A pure imaginary is a complex number whose square is negative, so 0 is not considered as a pure imaginary.

# Conjugate of a complex number

$z = a + bi$ is a complex number.
The conjugate of z is $\bar{z} = a - ib$

Example:

$z = 1 + 3i \qquad \bar{z} = 1 - 3i$

$z = -1 + 3i \qquad \bar{z} = -1 - 3i$

$z = 1 - 3i \qquad \bar{z} = 1 + 3i$

$z = -1 - 3i \qquad \bar{z} = -1 + 3i$

# Conjugate of a complex number

## Properties

❶ $\bar{\bar{z}} = z$

$\bar{z} = a - bi$

$\bar{\bar{z}} = \overline{a - bi} = a + bi = z$

# Conjugate of a complex number

## Properties

❷ $\overline{z+z'} = \bar{z} + \overline{z'}$

$$\overline{z+z'} = \overline{(a+a')+(b+b')i}$$

$$= (a+a') - (b+b')i$$

$$= a + a' - bi - b'i$$

$$= a - bi + a' - b'i = \bar{z} + \overline{z'}$$

# Conjugate of a complex number

## Properties

❷ $\overline{z+z'} = \bar{z} + \overline{z'}$

Example:

$z = 1 + 2i \quad ; \quad z' = 3 + 3i$

$\overline{z+z'} = \overline{4+5i} = 4 - 5i$

$\bar{z} + \overline{z'} = 1 - 2i + 3 - 3i = 4 - 5i$

# Conjugate of a complex number

## Properties

❸ $\overline{zz'} = \bar{z}\,\overline{z'}$

$$\overline{zz'} = \overline{aa' - bb' + (ab' + ba')i}$$

$$= aa' - bb' - (ab' + ba')i$$

$$= aa' - bb' - ab'i - ba'i$$

$$\bar{z}\overline{z'} = (a - bi)(a' - b'i)$$

$$= aa' - ab'i - ba'i + bb'i^2$$

$$= aa' - ab'i + ba'i - bb'$$

$$= \overline{zz'}$$

# Conjugate of a complex number

## Properties

❸ $\overline{zz'} = \bar{z}\,\overline{z'}$

Example:

$z = 1 + 2i \quad ; \quad z' = 3 + 3i$

$zz' = 1 \times 3 - 2 \times 3 + (1 \times 3 + 2 \times 3)i = -3 + 9i$

$\overline{zz'} = -3 - 9i$

$\bar{z}\overline{z'} = (1 - 2i)(3 - 3i) = 1 \times 3 - (-2)(-3) + \big(1 \times (-3) + (-2)(3)\big)i$

$= -3 - 9i$

# Conjugate of a complex number

## Properties

❹ $z\bar{z} = a^2 + b^2$

$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 + b^2$

Example:

$z = 1 + 2i$

$z\bar{z} = (1 + 2i)(1 - 2i) = 1^2 + 2^2 = 5$

$z = 3 + 3i$

$z\bar{z} = (3 + 3i)(3 - 3i) = 3^2 + 3^2 = 18$

# Application

Answer with true or false and justify.

1. $i^3 = i$

False since:

$i^3 = i \times i^2 = i(-1) = -i$

# Application

Answer with true or false and justify.

2. $z = (2i - 1)(2i + 1)$ is a real number.

True since:

$(2i - 1)(2i + 1) = (2i)^2 - 1^2 = 4i^2 - 1 = -4 - 1 = -5$ which is real

# Application

Answer with true or false and justify.

3. $z = (1+2i)^2 + (1-2i)^2$ is a pure imaginary

False since:

$z = (1+2i)^2 + (1-2i)^2 = 1 + 4i + 4i^2 + 1 - 4i + 4i^2$

$= 2 - 4 - 4 = -6$ which is real

# Application

Answer with true or false and justify.

4. The imaginary part of $z = \frac{2-i}{3} + i$ is $\frac{2}{3}$

True since:

$$z = \frac{2}{3} - \frac{i}{3} + i = \frac{2}{3} + i\left(-\frac{1}{3} + 1\right) = \frac{2}{3} + \frac{2}{3}i \text{ so } Im(z) = \frac{2}{3}$$

# Application

Answer with true or false and justify.

5. If $z = \frac{2}{3} i(i+2)^2$, then $Re(z) = 2$

False since:

$z = \frac{2}{3} i(i^2 + 4i + 4) = \frac{2}{3} i(-1 + 4i + 4) = \frac{2}{3} i(3 + 4i) = 2i + \frac{8}{3} i^2 = 2i - \frac{8}{3}$

So $Re(z) = -\frac{8}{3}$

# Application

Answer with true or false and justify.

6. If $Re\left(2x - i + \sqrt{3}\right) = 4$ , then $x = 0$

False since:

$z = 2x - i + \sqrt{3} = 2x + \sqrt{3} - i$

$Re(z) = 2x + \sqrt{3}$

So $2x + \sqrt{3} = 4$ ; $x = \frac{4-\sqrt{3}}{2} \neq 0$

# Application

Answer with true or false and justify.

7. The conjugate of $z = (3+2i)^2$ is $9-4i$

False since:

$z = (3+2i)^2 = 9 + 12i + 4i^2 = 9 + 12i - 4 = 5 + 12i$

$\bar{z} = 5 - 12i$